JENSEN: Probably the best inequality in the world

Paul Wilmott

Probably the best inequality in the world

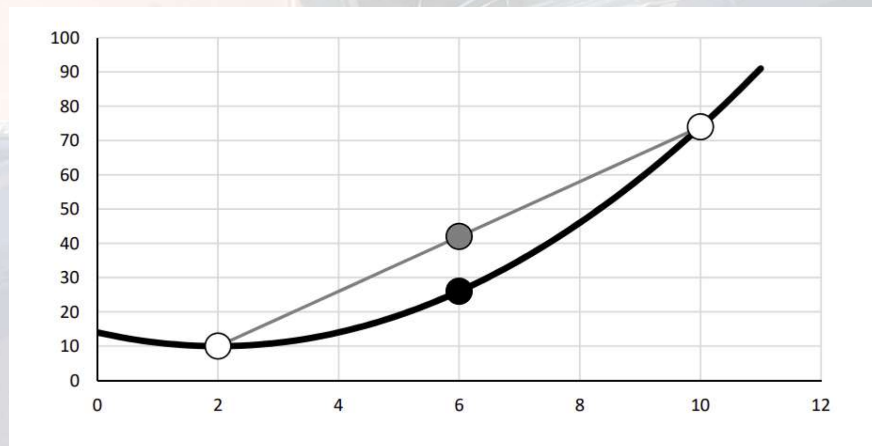
Johan Ludwig William Valdemar Jensen

- Jensen was born in Nakskov, Denmark. Although he studied mathematics among various subjects at college, and even published a research paper in mathematics, he learned advanced math topics later by himself and never held any academic position. Instead, he was a successful engineer for the Copenhagen Telephone Company between 1881 and 1924, and became head of the technical department in 1890. All his mathematics research was carried out in his spare time. Jensen is mostly renowned for his famous inequality, **Jensen's Inequality**.

Jensen’s Inequality in words

- Jensen’s Inequality concerns averages, or integrals, or sums of convex functions
- It comes in several forms (more later)
- One form is that the

  “***Average of a convex function at random points is greater than the function of the average of those points***”



Jensen everywhere

- Options
- Utility theory
- Taxes
- Referendums
- …

Jensen’s Inequality in math

- Discrete
- Continuous
- Probabilistic
- Deterministic
- …

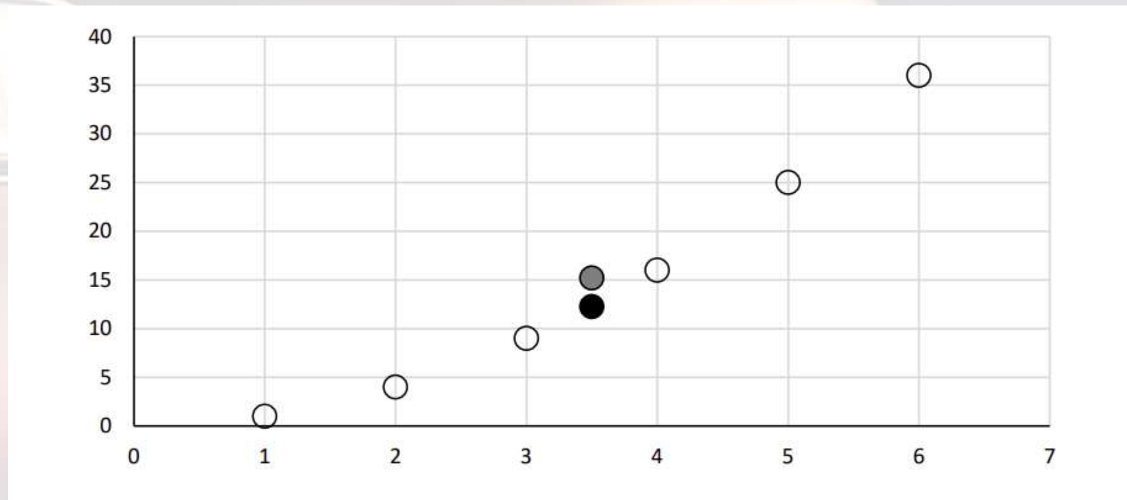
$$\frac{\sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} p_i} \geq f\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}\right)$$

$$\mathrm{E}[f(X)] \geq f\left(\mathrm{E}[X]\right)$$

$$\int p(x) f(x) dx \geq f\left(\int x p(x) dx\right)$$

Trivial example

- Dice game
- Win \$ spots
- Win \$ spots^2



A useful approximation

- Taylor series: $$f(x) = f(\bar{x}) + (x - \bar{x})\frac{df}{dx}\bigg|_{x=\bar{x}} + \frac{1}{2}(x - \bar{x})^2\frac{d^2 f}{dx^2}\bigg|_{x=\bar{x}} + \cdots$$
- $$\mathrm{E}[f(x)] = f(\mathrm{E}[x]) + \frac{1}{2}\,\mathrm{E}[(x - \mathrm{E}[x])^2]\,\frac{d^2 f}{dx^2}\bigg|_{x=\bar{x}} + \cdots$$
- $$\mathrm{E}[f(x)] - f(\mathrm{E}[x]) = \frac{1}{2}\,\mathrm{Var}[x]\,\frac{d^2 f}{dx^2}\bigg|_{x=\bar{x}} + \cdots$$

Jensen in option valuation

- Convexity!

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- Note, without the convexity term the solution for a call option would simply be

$$e^{-r(T-t)}\max(Se^{r(T-t)} - E, 0)$$

E.g. Convexity adjustments

- Whenever a contract is *not* linear in traded instrument(s)

$$E^{Q_S}[L(S,T)] = F(0;S,T) + \frac{\tau(S,T)\mathrm{Var}_{Q_T}(L(S,T))}{1+\tau(S,T)F(0;S,T)}$$

"LIBOR in arrears"

$$E^{Q_{T'}}[L(S,T)] = F(0;S,T)\left[1 + \frac{1-P(0,T)/P(0,T')}{F^2(0;S,T)}\mathrm{Var}_{Q_T}(L(S,T))\right], \forall T' \in [S,T]$$

"LIBOR paid at arbitrary time"

$$E^{Q_{T'}}\left[S_{\alpha,\beta}(T_\alpha)\right] = S_{\alpha,\beta}(0)\left[1 + \frac{1 - \frac{P(0,T_\alpha)-P(0,T_\beta)}{S_{\alpha,\beta}(0)P(0,T')\sum_{i=\alpha+1}^{\beta}\tau_i}}{S^2_{\alpha,\beta}(0)}\mathrm{Var}_N(S_{\alpha,\beta}(T_\alpha))\right]$$

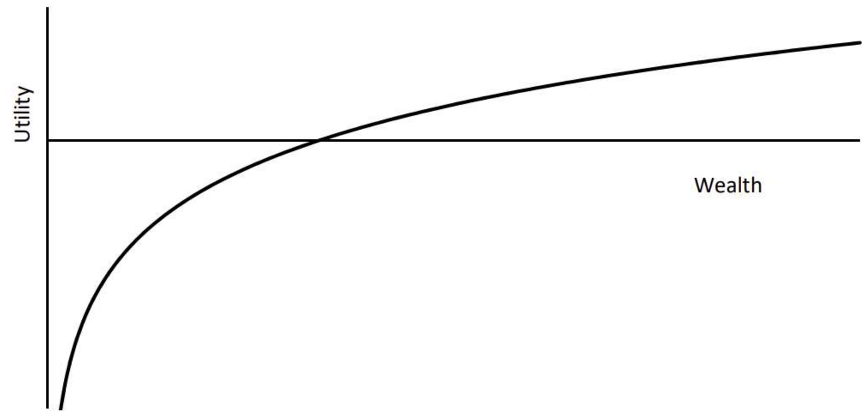
"Constant Maturity Swap"

- From "Convexity Adjustment: A User's Guide" by Yan Zeng

Utility Theory

- Within economics, the concept of utility is used to model worth or value … a utility function … represents a single consumer's preference ordering over a choice set but is not comparable across consumers. (Wikipedia)
- $U(W)$
- Coin toss, win or lose $1:

$$U(W_c) = \tfrac{1}{2}U(1) + \tfrac{1}{2}U(-1)$$



- Small bet approximation (Start with W):

$$W_c \sim W + \tfrac{1}{2}\epsilon^2 \frac{U''(W)}{U'(W)} + \cdots$$

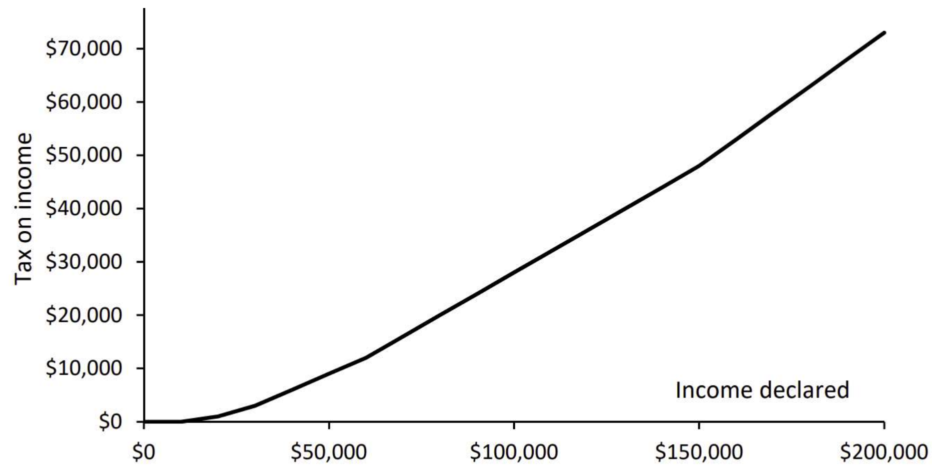
Taxation

- Salary is S, tax is $T(S)$
- Take-home pay $S - T(S)$
- Is it better to earn S for two years or zero one year and $2S$ the next?

$$2S - T(2S) - 2\big(S - T(S)\big) = 2T(S) - T(2S) < 0$$

(Assuming $T(0) = 0$!!! Wealth tax…arghhhhh!!!!)

- Convex tax encourages consistency
- Discourages starting a new business!



With added utility

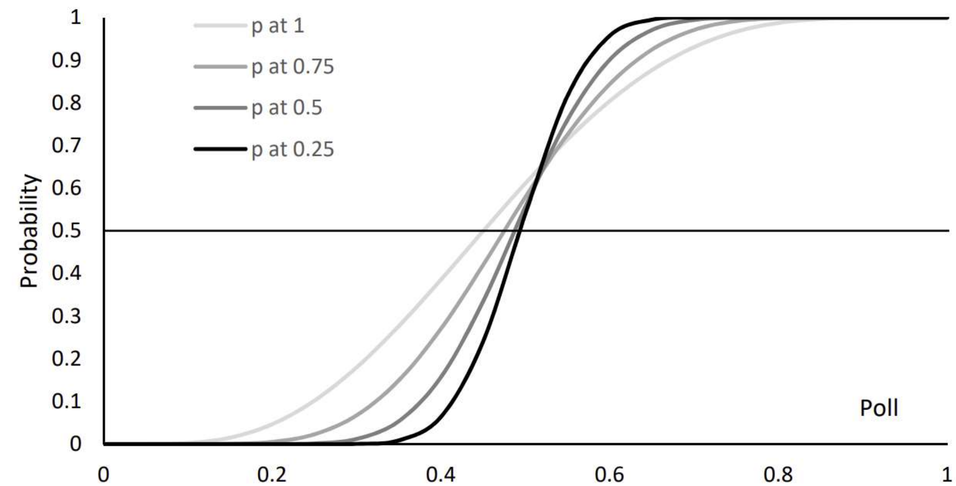
- Work with $U(S - T(S))$:

$$S_c = S + \frac{1}{2}\epsilon^2 \left(\frac{U''(1 - T')}{U'} - \frac{T''}{1 - T'} \right) + \cdots$$

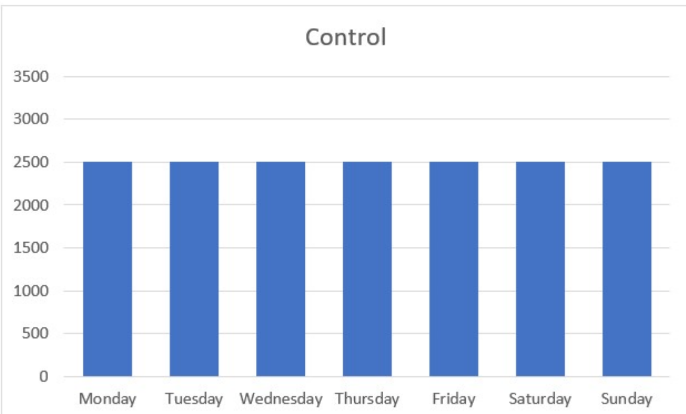
- Utility and tax both discourage entrepreneurship!
- To overcome disincentive of taxation you have to be risk seeking!
- Or…and I really like this idea…the marginal rate of tax should decrease as you earn more!!!

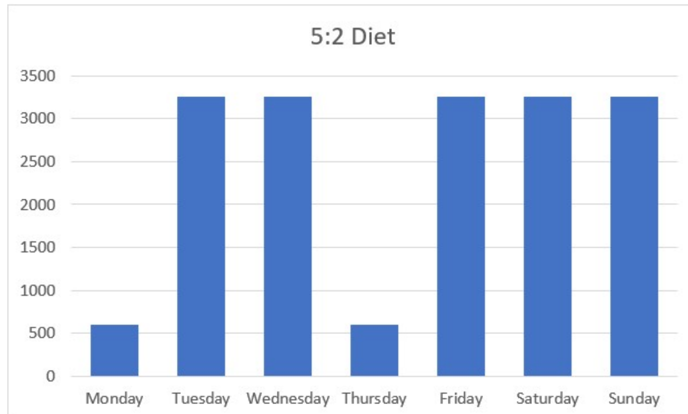
Referendums

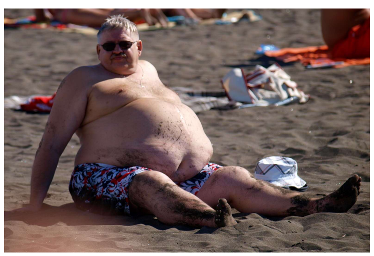
- Toy model
- Probability of winning a referendum against current (random) poll
- Sign of convexity
- When and where to “rock the boat”!
- (Observation, small sample size:
  Referendums get called when support is around 50%!)

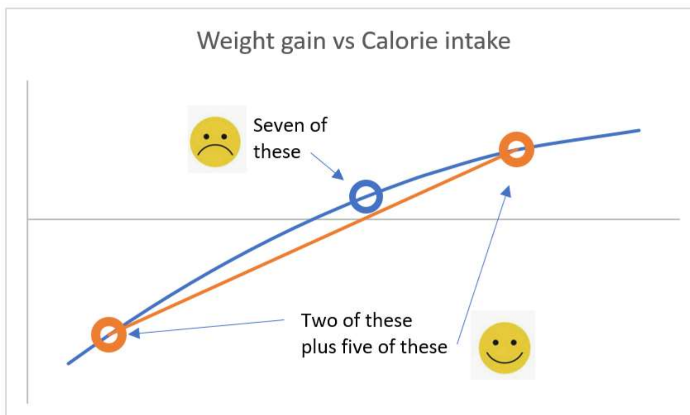


5:2 Diet!









"Nothing tastes as good as skinny feels."
Kate Moss

Add utility theory!!!

Everywhere!

- In the Wimbledon Men's Final 2019 Novak Djokovic played Roger Federer. Federer had more Aces, fewer faults, won more games, won more points (218 to 204), but lost the match.

- In the 2016 US election Hillary Clinton got 65,853,514 votes and Donald Trump 62,984,828. But Donald Trump became President thanks to winning more Electoral Votes.

- It's worth noting that people claimed Clinton won the election, but no one is claiming that Federer won 2019 Wimbledon!!!

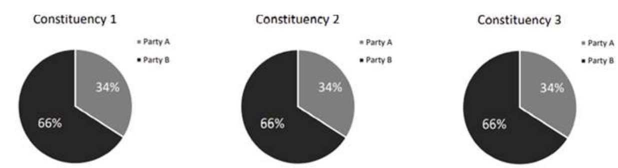


Figure 9.1: Three constituencies, Scenario 1.

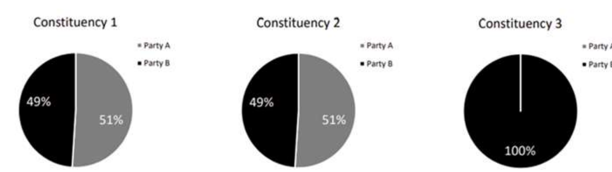


Figure 9.2: Three constituencies, Scenario 2.